PORTFOLIO INVESTMENT MODELS WITH ASYMMETRIC RISK MEASURES AND USING GENETIC ALGORITHMS

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ABSTRACT
This article is devoted to the use of alternative risk measures in the construction of portfolio investment models. We considered asymmetric, coherent and complex risk measures. We carried out a comparative analysis of the effectiveness of various metrics. We used genetic algorithms to solve the optimization problems. We carried out the experiments with data on assets that are traded on the Russian stock market. Based on the results of experiments with the construction of investment portfolios on the Russian stock market, we have concluded that the asymmetric risk measures considered more adequately characterize the risk concept in comparison with the classical methods, since they reflect the risk of portfolio yield change only in a negative direction. All portfolios have been built only with positive weights in the experiments conducted. It follows from the results of the experiments conducted that the portfolios in which the indicators of half-dispersion and CVaR have been used as a risk measure are the best ratio of risk and yield indicators. We have observed the results equivalent to using a standard risk measure - the mean square deviation - when using the mean absolute deviation (MAD) as a risk measure. The following fact should be considered for all models: they are all optimized on the historical data. On this basis we have concluded that it is necessary to periodically recalculate the portfolio weight after a short period. This point is especially relevant for the maximum drawdown risk indicator - CDaR. The issue of a time horizon for recalculations is worthy of a separate study and is still open.

Keywords: portfolio investment, asymmetric risk measures, genetic algorithms, optimization, mathematical methods.

INTRODUCTION
The choice of the optimal investment portfolio is a key task in the activities of commercial banks, pension funds, insurance companies, etc. The main goal of our study is to analyze the effectiveness of alternative risk measures in the tasks of choosing the optimal investment portfolio. Most of the risk measures considered in this article are asymmetric and, unlike symmetric measures, reflect the risk of price movement in the negative direction. The negative direction should mean a decrease in prices with long positions in the portfolio and an increase in prices with short positions in the portfolio.

TEXT OF ARTICLE
Let us define the main variables that will be used later. Let us assume that \( x = (x_1, x_2, \ldots, x_n) \) – the vector determining the investment portfolio structure, \( n \) – number of assets in the investment portfolio. Wherein:

\[
\sum_{i=1}^{n} x_i = e^T x = 1,
\]

where \( e \) – a single row vector \( 1 \times n \). The returns on assets are random variables and are characterized by a vector \( r = (r_1, r_2, \ldots, r_n) \). Obviously, the investment portfolio yield will be calculated as:
Sometimes, alternative measures are used instead of the mean yield: yield forecasts for time series models [1], growth potential assessment, yield estimation by Black-Litterman model [2]. In our study, we have used the Black-Litterman mode, since we consider it to be the most successful for application in practice [3]. $R_0$ and $Risk_0$ – the portfolio yield lower limit and the upper limit of a certain risk type, respectively. Often the benchmark yield $R_{BM}$ and the benchmark risk $Risk_{BM}$ are taken as these values. Different stock indices (MICEX index, RTS index) can act as a benchmark. Let us consider in more detail some of the risk measures.

*Value-At-Risk (VaR)*

The value $VaR$ reflects the amount of loss that corresponds to the lower quantile of the portfolio yield distribution for a given probability $\alpha$. This indicator was actively used in the financial risk management after the release of the international document Basel II [4]. In this document, $VaR$ is one of the recommended measures for the quantitative risk assessment. The value $VaR$ for the confidence level $\alpha$ should be calculated as follows:

$$VaR_\alpha(R) = -F^{-1}_R(\alpha)W,$$

where $F^{-1}_R$ – the cumulative value distribution function $R$, $W$ – the value of the investor's capital in monetary terms. From the point of view of probability theory $VaR - \alpha$-quantile of yield distribution. Thus, $VaR$ shows, that the portfolio loss value may exceed the value $VaR_\alpha(R)$ with a probability of $\alpha$. In other words, the portfolio loss will not exceed the value $VaR_\alpha(R)$ with a probability of $1 - \alpha$. It is usually used the value of $\alpha = 0.05$. The $VaR$ statistical evaluation, which was used in the optimization tasks, is calculated as follows:

$$VaR_\alpha(R) = r^T x - 1.645 \sqrt{Vx},$$

where $V$ – the matrix of covariance of historical returns, 1.645 - the quantile of normal distribution for the probability of $\alpha = 0.05$. $VaR$ value has a certain drawback - it does not satisfy the semi-additive property of a coherent [5] risk measure: the portfolio risk may exceed the sum of the risks of sub-portfolios or instruments.

*Conditional Value-at-Risk (CVaR)*

$CVaR$ is one of the areas of $VaR$ methodology development and reflects the mean value of losses that may be exceeded by $VaR$. Thus, $CVaR$ of $\alpha$-level is calculated as follows [5]:

$$CVaR_\alpha(f(r,x)) = M(f(r,x)|f(r,x) \geq VaR_\alpha(R)),$$

where $M(\cdot)$ denotes the mathematical expectation. The value of negative portfolio yield acts as a loss function, in the simplest case:

$$f(r,x) = -r^T x = -R.$$
This risk measure is coherent and allows more adequately assessing the risk of portfolios and individual assets that have a yield distribution with "heavy tails".

Mean Absolute Deviation, MAD

MAD indicator was firstly mentioned in the paper [6]. Back in 1991, it was developed the models for choosing the optimal investment portfolio, where the MAD value was the main risk measure. At the same time, it was assumed that the return on assets was normally distributed in the portfolio. The main model advantage based on this indicator is the absence of the need to calculate the covariance matrix of returns on assets. It has been shown in [7] that the models with MAD minimization give results close to the model results with minimized dispersion. The following formula is used to evaluate MAD:

\[
MAD(R) = M\left(\left|R - MR\right|\right).
\]  

(7)

The statistical MAD evaluation is as follows:

\[
\tilde{MAD}(R) = \frac{1}{N} \sum_{i=1}^{N} \left|R_i - \bar{R}\right|
\]  

(8)

where \( R_1, R_2, ..., R_N \) – the random value sample \( R \), and \( \bar{R} = \frac{1}{N} \sum_{i=1}^{N} R_i \).

Complex Risk Measures, CRM

The so-called complex risk measures were offered in the paper [8], which allowed combining the CVaR and MAD indicators. An example of one of these indicators is presented below:

\[
CRM(R) = CVaR_{\alpha}(R) - \beta MAD(R),
\]  

(9)

where the ratio \( \beta \geq 0 \) allows changing the weight of dispersion measure when calculating the complex risk measure. In our experiments \( \beta = 0.1 \).

Lower Partial Moments, LMP

The risk measure LMP is determined by two parameters: indicator or degree \( k \) and target yield \( \tau \). For a certain value \( R \), the value LMP is calculated as follows:

\[
LMP_{\tau,k} = M\left(\left(\tau - R\right)^k \mid R < \tau\right)P(R < \tau)^k.
\]  

(10)

If we take the values of \( \tau = MR, \ k = 1, \ k = 2 \), then we obtain the values of half-variation and half-dispersion, respectively:

\[
SD(R) = M\left(\left(R - MR\right) \mid R < MR\right)P(R < MR),
\]  

\[
SV(R) = M\left(\left(R - MR\right)^2 \mid R < MR\right)P(R < MR)^2.
\]  

(11)

The statistical estimates of \( SD(R) \) and \( SV(R) \) are calculated as follows:
A risk-free yield rate or zero yield are often used as the typical values $\tau$. These values mean the risk of loss relative to the alternative costs and the risk of absolute losses, respectively. In this case, the degree $k$ is a parameter of risk aversion. More detailed information about $LPM$ is given in [9].

**Loss-oriented Utility Function**

The classical model of portfolio optimization is focused on maximizing the utility function. The investor offers to maximize the utility $u(R)$ of the portfolio yield distribution:

$$M(u(R)) \rightarrow \max .$$

This approach is based on the work of Neumann and Morgenstern [10]. The Neumann-Morgenstern utility function has a quadratic form. However, the quadratic form is not the only form of the utility function. Paper [6] shows that the utility function $u(R)$ should have at least the following properties: $u' > 0, u'' < 0, u'''> 0$.

$$u(x) = -\exp(-x).$$

This is just a possible kind of utility function. Depending on the purpose, other functions that more fully reflect the risk can be compiled. Due to the utility function described above, it is reflected the negative deviations from the mean value:

$$M(u(R - MR)) \rightarrow \max .$$

The statistical evaluation of utility function $M(u(R))$ is as follows:

$$M(u(R - \mu)) = \frac{1}{T} \sum_{t=1}^{T} -\exp(-X_t + \mu).$$

**Conditional Drawdown-at-Risk, CDaR**

The possibility of using the amount of drawdown as a risk measure was shown for the first time in the paper of Chekhlov, Ursaev and Zabarankin from the University of Florida, "Portfolio optimization with drawdown constraints" [7]. Drawdown is a decrease in the portfolio value relative to the historical maximum of value. In this case, the drawdown function looks like this:

$$f(x, j) = \max_{1 < k < j} \left\{ \sum_{i=1}^{n} \left[ 1 + \sum_{r=1}^{k} r_{ii} \right] x_{i} \right\} - \left\{ \sum_{i=1}^{n} \left[ 1 + \sum_{r=1}^{j} r_{ii} \right] x_{i} \right\}.$$ 

CDaR value is the expected drawdown value $(1 - \alpha)$ and can be calculated using the following formula:

$$\tilde{SD}(R) = \frac{1}{T} \sum_{t=1}^{T} \min \left[ (R_t - \bar{R}), 0 \right],$$

$$\tilde{SV}(R) = \frac{1}{T} \sum_{t=1}^{T} \min \left[ (R_t - \bar{R}), 0 \right]^2.$$
where $\eta$ – the drawdown threshold, that is, the portfolio depreciation value, the excess of which can be considered a drawdown. Table 1 summarizes the risk models examined.

**Table 1. Some asymmetric risk measures**

<table>
<thead>
<tr>
<th>#</th>
<th>Name</th>
<th>Formula Risk(R)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Value-at-Risk (VaR)</td>
<td>$VaR_\alpha^\cdot(R) = -F^{-1}_R(\alpha)W$</td>
</tr>
<tr>
<td>2</td>
<td>Conditional Value-at-Risk (CVaR)</td>
<td>$CVaR_\alpha^\cdot(f(r,x)) = M\left(f(r,x)</td>
</tr>
<tr>
<td>3</td>
<td>Mean Absolute Deviation (MAD)</td>
<td>$MAD(R) = M\left(</td>
</tr>
<tr>
<td>4</td>
<td>Complex Risk Measure (CRM)</td>
<td>$CRM(R) = CVaR_\alpha^\cdot(R) - \beta MAD(R)$</td>
</tr>
<tr>
<td>5</td>
<td>Lower partial moments (LMP)</td>
<td>$LMP_{\tau,k} = M\left(\left(\tau - R\right)^k</td>
</tr>
<tr>
<td>6</td>
<td>Half-dispersion (LMP at $k=2$)</td>
<td>$SV(R) = M\left(\left(R - MR\right)^2</td>
</tr>
<tr>
<td>7</td>
<td>Loss-oriented utility function</td>
<td>$M\left(u(R - MR)\right)$</td>
</tr>
<tr>
<td>8</td>
<td>Conditional Drawdown-at-Risk (CDaR)</td>
<td>$CDaR_\alpha^\cdot(x,\eta) = \eta + (1 - \alpha)^{-1}\sum_{j=1}^{s}\left[f(x,j) - \eta\right]^+$</td>
</tr>
</tbody>
</table>

**General model view**

Within the framework of the definitions considered, the following types of tasks can be solved: risk minimization at a given lower yield limit:

\[
Risk(R) \rightarrow \min \left\{ \begin{array}{l}
  r^T x \geq R_0 \\
  e^T x = 1 \\
  x \geq 0
\end{array} \right. \quad (19)
\]

and yield maximization at a given risk limit:

\[
r^T x \rightarrow \max \left\{ \begin{array}{l}
  Risk(R) \leq Risk_0 \\
  e^T x = 1 \\
  x \geq 0
\end{array} \right. \quad (20)
\]
Genetic algorithms

Solution of the optimization tasks (17) and (18) with the help of classical methods is rather laborious. If it is necessary to solve a large number of optimization tasks in a short time, it is convenient to use the genetic algorithms. The genetic algorithms are understood as the heuristic search algorithms based on the evolution of approximations used for optimization and modeling tasks [11, 12]. The scheme of the simple genetic algorithm is shown in Figure 1.

![Diagram of a simple genetic algorithm]

<table>
<thead>
<tr>
<th>Начало алгоритма</th>
<th>Start of the algorithm</th>
</tr>
</thead>
<tbody>
<tr>
<td>Создание стартовой популяции</td>
<td>Creating a starting population</td>
</tr>
<tr>
<td>Пока не найдено решение или не выполнен критерий остановки</td>
<td>A solution has not been found or a stopping criterion has not been performed yet</td>
</tr>
<tr>
<td>Скрещивание</td>
<td>Crossing</td>
</tr>
<tr>
<td>Мутация</td>
<td>Mutation</td>
</tr>
<tr>
<td>Отбор</td>
<td>Selection</td>
</tr>
<tr>
<td>Завершение алгоритма</td>
<td>Completion of the algorithm</td>
</tr>
</tbody>
</table>

**Figure 1.** A block diagram of a simple genetic algorithm
Due to the genetic algorithms it is possible to solve complex nonlinear optimization tasks with constraints. Table 2 shows the values of algorithm parameters that have been used in solving the optimization tasks.

<table>
<thead>
<tr>
<th>#</th>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Number of chromosomes</td>
<td>7</td>
</tr>
<tr>
<td>2</td>
<td>Number of individuals</td>
<td>1,000</td>
</tr>
<tr>
<td>3</td>
<td>Probability of crossing</td>
<td>0.8</td>
</tr>
<tr>
<td>4</td>
<td>Probability of mutation</td>
<td>0.2</td>
</tr>
<tr>
<td>5</td>
<td>Selection type</td>
<td>Tournament</td>
</tr>
<tr>
<td>6</td>
<td>Crossing type</td>
<td>Bipunctate</td>
</tr>
<tr>
<td>7</td>
<td>Type of the new generation formation</td>
<td>Only descendants</td>
</tr>
</tbody>
</table>

**Table 2.** Values of genetic algorithm parameters

**Experiment results**

We conducted the experiments with models for choosing the optimal investment portfolio, where the measures considered in this article were used as the risk measures. The experiments were carried out with some highly liquid assets traded on the Moscow Interbank Currency Exchange (common shares of Tatneft OJSC (TATN), Gazprom OJSC (GAZP), Rostelecom OJSC (RTKM), VTB OJSC (VTB), Lukoil OJSC (LKOH), Polyus Gold OJSC (PLZL), Sberbank OJSC (SBER)). The portfolio structures for different values of the corresponding risk measures are shown in Figures 2-8. The Sharpe ratio values are marked along the abscissa in these figures [1].

**Figure 2.** Model with VaR risk measure

**Figure 3.** Model with CVaR risk measure
Figure 4. Model with MAD risk measure

Figure 5. Model with CRM risk measure

Figure 6. Model with SD risk measure

Figure 7. Model with risk measure as a utility function

Figure 8. Model with CDaR risk measure
Let us perform a comparative analysis of various risk measures with a standard risk measure - the mean square yield deviation. For comparison, the mean values of each of the indicators considered were taken. The relevant information is specified in Table 3.

**Table 3. Comparison of different risk measures**

<table>
<thead>
<tr>
<th>#</th>
<th>Risk indicator</th>
<th>The value of risk indicator for a portfolio optimized under the mean square deviation</th>
<th>The value of risk indicator for an optimized portfolio</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>VaR</td>
<td>0.20</td>
<td>0.21</td>
</tr>
<tr>
<td>2</td>
<td>CVaR</td>
<td>0.26</td>
<td>0.25</td>
</tr>
<tr>
<td>3</td>
<td>MAD</td>
<td>1.61</td>
<td>1.56</td>
</tr>
<tr>
<td>4</td>
<td>CRM</td>
<td>1.11</td>
<td>1.28</td>
</tr>
<tr>
<td>5</td>
<td>SD</td>
<td>0.71</td>
<td>0.67</td>
</tr>
<tr>
<td>6</td>
<td>U</td>
<td>2.35</td>
<td>2.77</td>
</tr>
<tr>
<td>7</td>
<td>MDD</td>
<td>14.71</td>
<td>12.5</td>
</tr>
</tbody>
</table>

Having analyzed the table, we can see that the alternative risk measures, for the most part, show better results than the standard risk measure - the mean square deviation. Let us also consider the average value of the Sharpe ratio for portfolios optimized for each of the risk measures, including for the mean square deviation. The results are presented in Table 4.

**Table 4. Mean value of the Sharpe ratio for various portfolios**

<table>
<thead>
<tr>
<th>#</th>
<th>Name</th>
<th>Mean value of the Sharpe ratio</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Value-at-Risk (VaR)</td>
<td>1.25</td>
</tr>
<tr>
<td>2</td>
<td>Conditional Value-at-Risk (CVaR)</td>
<td>1.27</td>
</tr>
<tr>
<td>3</td>
<td>Mean Absolute Deviation (MAD)</td>
<td>1.31</td>
</tr>
<tr>
<td>4</td>
<td>Complex Risk Measure (CMR)</td>
<td>1.20</td>
</tr>
<tr>
<td>5</td>
<td>Half-dispersion (LMP at k=2)</td>
<td>1.30</td>
</tr>
<tr>
<td>6</td>
<td>Loss-oriented utility function</td>
<td>1.28</td>
</tr>
<tr>
<td>7</td>
<td>Conditional Drawdown-at-Risk (CDaR)</td>
<td>1.23</td>
</tr>
</tbody>
</table>

We can see again that the alternative risk measures, for the most part, show better results than the standard risk measure - the mean square deviation.
METHODS
During the study, the authors used the following methods:

1. Selective analysis of specialized literature with a high citation index for the subject matter indicated in the title of this article. In particular, we considered the following risk assessment measures: Value-At-Risk (VaR), Conditional Value-at-Risk (CVaR), Mean Absolute Deviation (MAD), Complex Risk Measures (CRM), Lower Partial Moments $LMP$, Loss-Oriented Utility Function, Conditional Drawdown-at-Risk (CDaR), as well as the theoretical foundations of genetic algorithms.

2. We carried out a comparative analysis of the collected information on the criteria determined by the authors with the purpose of revealing the merits, shortcomings of the methods considered and assessing the possibility of their practical application.

3. The study results were given the author's interpretation, and the appropriate conclusions were drawn.

RESULTS
Based on the results of experiments with the construction of investment portfolios on the Russian stock market, we may conclude that the asymmetric risk measures considered more adequately characterize the risk concept in comparison with the classical methods, since they reflect the risk of portfolio yield change only in a negative direction. The negative direction should mean a decrease in prices with long positions in the portfolio and an increase in prices with short positions in the portfolio. All portfolios have been built only with positive weights in our experiments. It follows from the results of our experiments that the portfolios in which the indicators of half-dispersion and $CVaR$ have been used as a risk measure are the best ratio of risk and yield indicators.

DISCUSSION
We have observed the results equivalent to using a standard risk measure - the mean square deviation - when using the mean absolute deviation ($MAD$) as a risk measure. The results for a complex $CRM$ risk measure depend significantly on the ratio $\beta$. With a chosen ratio $\beta$, the portfolios optimized under $CRM$ showed worse results compared to $CVaR$ measure. The same can be said about the utility function as a risk measure. The final result will essentially depend on the utility function chosen by the investor.

CONCLUSIONS
The following fact should be considered for all models: they are all optimized on the historical data. Therefore, it is necessary to periodically recalculate the portfolio weight after a short period. This point is especially relevant for the maximum drawdown risk indicator - $CDaR$. The issue of a time horizon for recalculation is worthy of a separate study and is still open. We should also note the convenience of genetic algorithms in solving a large number of optimization problems with complex constraints.

ACKNOWLEDGEMENTS
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